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Nonlinear squeezed states for SU(1,1) Lie algebra

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Abstract. The nonlinear extensions of the single-mode squeezed vacuum and squeezed coherent states are studied. We have constructed the nonlinear squeezed states (NLSS's) realization of SU(1,1) Lie algebra. Two cases of this realization are considered for unitary and non-unitary deformation operator function. The nonlinear squeezed coherent states (NLSCS's) are defined and special cases of these states are obtained. Some nonclassical properties of these states are discussed. The s-parameterized characteristic function and various moments are calculated. The Glauber second-order coherence function is calculated. The squeezing properties of the NLSCS's are studied. Analytical and numerical results for the quadrature component distributions for the NLSCS's are presented. A generation scheme for NLSCS's using the trapped ions centre-of-mass motion approach is proposed.

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1 Introduction

Coherent states (CS's) of a simple harmonic oscillator have considerable applications in the field of quantum optics. It satisfies the eigenvalue equation $a|\alpha\rangle = \alpha|\alpha\rangle$, with $\alpha = |\alpha|\exp(i\theta)$ and a is the annihilation operator for bosons. However, the coherent state $|\alpha\rangle$ parameterized by α can be cast as the result of the action of the displacement operator $D(\alpha)$ on the ground state $|0\rangle$, with $D(\alpha) = \exp(\alpha a^+ - \alpha^* a)$ [1], with a^+ the creation operator for bosons being the Hermitian conjugate of a, and α^* the complex conjugate of α. In addition to CS's, squeezed states (SS's) are becoming increasingly important, these are the non-classical states of the electromagnetic field in which certain observables exhibit fluctuations less than in the vacuum state [2]. These states are very useful in various branches of physics.

On the other hand, considerable attention has been paid to the deformation of the harmonic oscillator algebra of creation and annihilation operators [3]. Some important physical concepts such as the CS's, the even- and odd-coherent states for ordinary harmonic oscillator have been extended to deformation case. The nonlinear coherent states (NLCS's) $|\alpha\rangle_f$, are right-hand eigenstates of the product of the boson annihilation operator a and nonlinear function $f(a^+a)$ of the number operator $N = a^+a$, i.e., they satisfy $af(N)|\alpha\rangle_f = \alpha|\alpha\rangle_f$. The nature of the nonlinearity depends on the choice of the function f. These states may appear as stationary states of the center-of-mass motion of a trapped and bichromatically laser-driven ion far from the Lamb-Dicke regime [4]. These states as well as their superpositions have been introduced and studied [5]. These NLCS's exhibit nonclassical features like squeezing and self-splitting [5]. The construction of nonlinear squeezed states (NLSS's) have been given in reference [6].

The SU(1,1) Lie algebra is of great interest in quantum optics because it can characterize many kinds of quantum optical systems [7–14]. It has recently been used by many researchers to investigate the nonclassical properties of light in quantum optical systems [12–14]. In particular, the bosonic realization of SU(1,1) describes the degenerate and non-degenerate parametric amplifiers [7,12]. The squeezed states of photons have been considered in terms of SU(1,1) Lie algebra and the coherent states associated with this algebra. The squeezed vacuum state is a special case of the Perelomov SU(1,1) coherent state [7–13].

Nonclassical effects are characterized by photon antibunching, sub-Poissonian photon statistics and quadrature squeezing [15]. The definition of nonclassicality is based on the existence of a well- behaved P-function [1]. This means that a state is considered to have a classical counter part if the P -function has the properties of a probability measure. For a nonclassical state it fails to be interpreted as a probability and it may have singularities (Titulaer and Glauber in [1]). Some methods for the characterizations of the nonclassical properties of radiation

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related to the P-function have also been discussed [16–18]. Also, the negativity of the Wigner function may be used as a signature of nonclassicality [19].

A nonclassical state has been generated experimentally by applying a series of laser pulses on a laser-cooled trapped ion in its motional ground state [20]. Hence the ion trap is a realization of the harmonic oscillator model in quantum mechanics. In this manner, nonclassical states of the atomic motion and entangled states of internal and external degrees of freedom of the atom could be realized [21–23]. The engineering of the vibronic coupling of trapped atom by appropriate laser excitations have been proposed [21]. The realization of special classes of NLCS, corresponding to special choices of the function $f(n)$ have been investigated [20–23]. Here we propose an experimental scheme to engineer the NLSCS's.

It is well known that there are three definitions of SS's and CS's [1,2,13]: (1) displacement operator acting on vacuum states, (2) eigenstates of the linear combination of creation and annihilation operators, and (3) minimum uncertainty states. Therefore, the SS's are defined to be the eigenstates of the operator b with eigenvalues α, [2]

$$b|\alpha,\mu,\nu\rangle = \alpha|\alpha,\mu,\nu\rangle \tag{1}$$

where μ and ν are complex numbers satisfying $|\mu|^2 - |\nu|^2 = 1$ and $b = \mu a + \nu a^+$ and its adjoint b^+, thus it follows $a = \mu^* b - \nu b^+$. For $\nu = 0$, then $|\alpha,\mu,\nu\rangle$ becomes the ordinary coherent state. Also, the squeezed state can be defined by the action of a squeeze unitary operator [2] on the coherent state namely $|z,\alpha\rangle = S(z)|\alpha\rangle$, with

$$S(z) = \exp\left[\frac{1}{2}(z^* a^2 - z a^{+2})\right] \tag{2}$$

where $z = r\exp(i\phi)$ is a complex number. We easily obtain

$$S(z)|\alpha\rangle = S(z)D(\alpha)|0\rangle = D(\alpha_0)S(z)|0\rangle \tag{3}$$

where the parameters α_0 and $\alpha_0{}^*$ represent equivalent shift of a squeezed vacuum as $\alpha = \mu\alpha_0 + \nu\alpha_0{}^*$. And z, z^* are related to μ and ν by $\mu = \cosh r$, and $\nu = \exp(i\phi)\sinh r$.

The aim of this work is to construct the NLSS's as realizations of SU(1,1) Lie algebra, define the nonlinear squeezed coherent states (NLSCS's) and study some of their statistical properties. One problem has been that: it is relatively easy to define states of the field with nonclassical properties but it is another matter altogether to find suitable mechanisms for their generation. Therefore, we use vibrational motion of trapped ions to realize the NLSCS's. This paper is organized as follows. In Section 2 we briefly discuss the construction of the NLSS's realization of SU(1,1) Lie algebra. In Section 3 we introduce the definition of NLSCS's and special cases are found. In Section 4 we discuss the statistical properties of NLSCS's such as, s-parameterized characteristic function (CF), moments and squeezing. Also, we discuss the quadrature component distributions for these states. A generation scheme for NLSCS's using the trapped ions centre-of-mass motion approach is introduced in Section 5.

2 NLSS's realization of SU(1,1) Lie algebra

The dynamical group SU(1,1) has long been used in quantum optics as it is intimately related to the squeeze operator which is an element of the SU(1,1) group [7]. This group is the simplest non-Abelian noncompact Lie group with a simple Lie algebra, and shares with SU(2) a common complex extension. The CS's of the SU(1,1) group can be divided into two broad categories: (a) the Barut-Girardello (BGCS's) [9] and (b) the Perelomov (PCS's) [10]. These two papers are the basic investigations in which the concept of CS's was extended beyond the Heisenberg-Weyl group for the first time. Perelomov [10] has elaborated and generalized the idea of CS's to other Lie groups, his book group methods were employed to study the properties of these systems. The (BGCS's) have been investigated in mathematical framework in [11]. The duality of these two types of SU(1,1) CS's and an intermediate type have been considered in reference [8].

The physical quantities observed experimentally in many optical effects based on emission and absorption photons can be associated with the creation (a^+) and annihilation (a) operators. Optical effects connected with the two-photon physics, are often related to the SU(1,1) Lie group [7–14]. it has been shown that the single- and two-mode bosonic realizations of the SU(1,1) Lie algebra have immediate relevance to the nonclassical squeezing properties of light [7–14]. The SU(1,1) Lie algebra is spanned by the three generators K_1, K_2, K_3,

$$[K_1, K_2] = -iK_3, \quad [K_2, K_3] = iK_1, \quad [K_3, K_1] = iK_2. \tag{4}$$

It is convenient to use the raising and lowering generators $K_\pm = K_1 \pm iK_2$, which satisfy

$$[K_3, K_\pm] = \pm K_\pm, \qquad [K_-, K_+] = 2K_3. \tag{5}$$

For any irreducible representation of SU(1,1) the Casimir operator $K^2 = K_3^2 - K_1^2 - K_2^2$ has the form $K^2 = k(k-1)I$. Thus a representation of SU(1,1) is determined by the parameter k.

2.1 The standard states

First we briefly review the standard case. The squeezed vacuum realization of the SU(1,1) Lie group is considered by taking the Ks operators in the form

$$K_+ = \frac{1}{2}a^{+2}, \quad K_- = \frac{1}{2}a^2, \quad K_3 = \frac{1}{2}\left(N + \frac{1}{2}\right) \tag{6}$$

with $N = a^+ a$ the photon number operator. The Casimir operator in this case becomes $K^2 = -3/16$. Therefore, there are two irreducible representations with $k = 1/4$ and $k = 3/4$ [7,13]. The state space associated with $k = 1/4$ is the even Fock sub-space with the orthonormal basis consisting of the even number eigenstates $\{|2n\rangle\}$. While the state space associated with $k = 3/4$ is the odd Fock subspace with the orthonormal basis $\{|2n+1\rangle\}$. The squeeze operator

$$S(z) = \exp(zK_+ - z^*K_-) = \exp\left(\frac{1}{2}z{a^+}^2 - \frac{1}{2}z^*a^2\right) \quad (7)$$

is the unitary group operator for the two-photon realization where K_+, K_-, K_3, are given by (6). In this case, the SU(1,1) coherent states are the single-mode squeezed states. For $k = 1/4$, the squeezed vacuum is given by

$$|\xi, 1/4\rangle = S(z)|0\rangle = (1 - |\xi|^2)^{1/4} \sum_{n=0}^{\infty} \frac{\sqrt{(2n)!}}{2^n n!} \xi^n |2n\rangle \quad (8a)$$

and for $k = 3/4$, the squeezed one-photon state is given by

$$|\xi, 3/4\rangle = S(z)|1\rangle = (1-|\xi|^2)^{3/4} \sum_{n=0}^{\infty} \frac{\sqrt{(2n+1)!}}{2^n n!} \xi^n |2n+1\rangle \quad (8b)$$

with $\xi = (z/|z|)\tanh|z| = e^{i\phi}\tanh r$.

2.2 The nonlinear realization

In what follows we mention the NLSS's realization of the SU(1,1) group by constructing the $K-$ operators in the following way [14],

$$K_+ = \frac{1}{2}(f(N)^+a^+)^2 = \frac{1}{2}{A^+}^2, \;\; K_- = \frac{1}{2}(af(N))^2 = \frac{1}{2}A^2, \quad (9a)$$

where the operator valued function $f(N)$ is a reasonably behaved function of the photon number operator N. For the operator K_3 to be in the form of (6) then f must be a unitary operator $f^+ = f^{-1}$. Under this condition

$$K_3 = \frac{1}{2}\left(N + \frac{1}{2}\right). \quad (9b)$$

The unitary group operator $S_f(z)$ for the nonlinear squeezing is the operator given by (7) but with $K_\pm, K_3$ given by (9) under the condition $f^+ = f^{-1}$. Therefore the SU(1,1) coherent states are the NLSS's. Consequently the non-linear squeezed vacuum is given by

$$|\xi, 1/4\rangle_f = (1 - |\xi|^2)^{1/4} \sum_{n=0}^{\infty} \frac{\sqrt{(2n)!}(f(2n)!)^{-1}}{2^n n!} \xi^n |2n\rangle \quad (10a)$$

while the nonlinear squeezed one-photon state is given by

$$|\xi, 3/4\rangle_f = (1 - |\xi|^2)^{3/4} \sum_{n=0}^{\infty} \frac{\sqrt{(2n+1)!}}{2^n n!(f(2n+1)!)} \xi^n |2n+1\rangle \quad (10b)$$

where $f(0) = 1$, $f(n)! = \prod_{i=0}^{n} f(i)$. The states (10) are the SU(1,1) group realization by a nonlinear squeezed vacuum and one-photon states.

2.3 Realization for non-unitary f

Even if the operator function $f(N)$ is not a unitary operator one can still define a NLSS's as given in [14]. The steps towards this depends on using a canonical conjugate operator. If we have

$$A = (af(N)), \qquad A^+ = ([f(N)]^+a^+), \quad (11a)$$

then the canonical conjugate operators are

$$B^+ = \frac{1}{f(N)}a^+, \qquad B = a\frac{1}{[f(N)]^+}. \quad (11b)$$

The operators A and B satisfy the commutation relations

$$[A, B^+] = 1, \qquad [B, A^+] = 1. \quad (11c)$$

In what follows the operator valued function f is assumed to be a well-behaved real function. The use of the operators A and B^+ (instead of A^+) does not insure the operator S being unitary, thus one looks for the eigenfunctions of the operator

$$C_1 = \frac{1}{\sqrt{1 - |\xi_1|^2}}(A - \xi_1 B^+)$$
$$\text{or} \quad C_2 = \frac{1}{\sqrt{1 - |\xi_2|^2}}(B - \xi_2 A^+) \quad (12)$$

with the eigenvalue zero, i.e., the nonlinear squeezed vacuum states are the solutions of the equations

$$C_1|\Psi_1\rangle_f = 0 \qquad \text{or} \qquad C_2|\Psi_2\rangle_f = 0. \quad (13)$$

It is straightforward to find the expression

$$|\Psi_1\rangle_f = N_1 \sum_{m=0}^{\infty} \frac{\sqrt{(2m)!}(f(2m)!)^{-1}}{2^m m!} \xi_1^m |2m\rangle \quad (14a)$$

and

$$|\Psi_2\rangle_f = N_2 \sum_{m=0}^{\infty} \frac{\sqrt{(2m)!}(f(2m)!)}{2^m m!} \xi_2^m |2m\rangle \quad (14b)$$

where N_1, N_2 are the normalization constants. While the nonlinear squeezed one-photon states are the solutions of the eigenvalue equations

$$C_i^2|\Phi_i\rangle = 0, \qquad i = 1, 2. \quad (15)$$

Carrying out the calculations, it is easy to find that these states are composed of the odd Fock states and are given by

$$|\Phi_1\rangle_f = \acute{N}_1 \sum_{m=0}^{\infty} \frac{\sqrt{(2m+1)!}(f(2m+1)!)^{-1}}{2^m m!} \xi_1^m |2m+1\rangle \quad (16a)$$

and

$$|\Phi_2\rangle_f = \acute{N}_2 \sum_{m=0}^{\infty} \frac{\sqrt{(2m+1)!}(f(2m+1)!)}{2^m m!} \xi_2^m |2m+1\rangle \quad (16b)$$

with $\acute{N}_i$ normalization constants. Equations (14) and (16) are formally similar to equations (10) which are the squeezed states and NLSS's realizations of the SU(1,1) group for the different Bargmann numbers.

The functions $|\Psi_i\rangle$ and $|\Phi_i\rangle$ may be formulated as results of applications of exponential operators on the states $|0\rangle$ and $|1\rangle$. In effect it is easy to find that

$$|\Psi_1\rangle = N_1 e^{\frac{1}{2}\xi_1 B^{+2}}|0\rangle, \qquad |\Phi_1\rangle = \acute{N}_1 e^{\frac{1}{2}\xi_1 B^{+2}}|1\rangle \qquad (17)$$

while for $|\Psi_2\rangle$ and $|\Phi_2\rangle$ B^+ is replaced by A^+. Some of the nonclassical properties of these states have been discussed recently [14].

3 Nonlinear squeezed coherent states

In this section we extend the investigation to the nonlinear squeezed states. We first start with the case when the operator valued function f is unitary.

3.1 The definition for unitary f

For the operator function $f(N)$ is unitary operator, i.e., $f^+ = f^{-1}$, and $[A, A^+] = 1$. Then the NLCS's are defined in the form [6]:

$$|\alpha\rangle_f = \exp(\alpha A^+ - \alpha^* A)|0\rangle = D_f(\alpha)|0\rangle, \qquad (18)$$

with α a complex number. The nonlinear squeezed vacuum states [6] are given by

$$|z\rangle_f = \exp\left[\frac{1}{2}(zA^{+2} - z^* A^2)\right]|0\rangle = S_f(z)|0\rangle \qquad (19)$$

and the nonlinear squeezed coherent states in this case is defined by

$$|z,\alpha\rangle_f = S_f(z)D_f(\alpha)|0\rangle \qquad (20)$$

which can be written as

$$|\xi,\beta\rangle_f = c_0 \sum_{m=0}^{\infty} \frac{1}{f(m)!\sqrt{m!}}\left(-\frac{1}{2}\xi\right)^{m/2} \times H_m\left[\frac{\beta\sqrt{1-|\xi|^2}}{\sqrt{-2\xi}}\right]|m\rangle. \qquad (21)$$

By using the Hermite relation [24],

$$\sum_{n=0}^{\infty} \frac{(t/2)^n}{n!} H_n(x)H_n(y) = (1-t^2)^{-1/2} \exp\left(\frac{2xyt - (x^2+y^2)t^2}{1-t^2}\right) \qquad (22a)$$

where $H_n(x)$ stands for the Hermite polynomials, given by

$$H_n(x) = \sum_{i=0}^{[\frac{n}{2}]} \frac{n!(-1)^i(2x)^{n-2i}}{i!(n-2i)!}. \qquad (22b)$$

Due to the fact that f is unitary, the normalization constant c_0 is the same as for the standard squeezed state $f=1$, in the form

$$|c_0|^{-2} = \frac{1}{\sqrt{1-|\xi|^2}} \exp\left\{-|\beta|^2 + \frac{|\xi|}{2}(\beta^2 e^{i\phi} + \beta^{*2} e^{-i\phi})\right\}. \qquad (23)$$

When we use the relation $\xi = -e^{i\phi}\tanh r$, or $\xi = -\nu/\mu$, and write $\beta = \mu\alpha_0 - \nu\alpha_0^* = \alpha$ in equation (3), the exact formula of Yeun's work [2] is retained. For the results of NLCS's we put $\xi = 0$ in equations (21) and (22). The coherent states normalization constant can be found when $\xi = 0$ in equation (23).

For the unitary deformation operator function, i.e., $ff^+ = I$, it is clear that the overcompleteness relation holds

$$\frac{1}{\pi}\int |\xi,\beta\rangle_{f\,f}\langle\beta,\xi|\, d^2\beta = 1. \qquad (24)$$

The overcompleteness relation is difficult to prove for the non-unitary case as we mention below.

3.2 The definition for non-unitary f

In this case, we define the nonlinear squeezed coherent states (NLSCS's) $|\xi,\beta\rangle_f$ as the eigenfunctions of the operators C_i of equation (12) such that

$$C_i|\xi_i,\beta_i\rangle_f = \beta_i|\xi_i,\beta_i\rangle_f \qquad i = 1,2 \qquad (25)$$

where β_i are complex numbers, and C_i given in equation (12). We next determine the solution to the eigenvalue equation (25). The NLSCS's are expressed in terms of the Fock states in the form

$$|\xi_i,\beta_i\rangle_f = \sum_{n=0}^{\infty} c_n|n\rangle. \qquad (26)$$

Substituting equation (26) into equation (25), we find the recursion relation among the coefficients c_n's

$$\sqrt{1-|\xi|^2}\beta c_m = \sqrt{m+1}f(m+1)c_{m+1} - \frac{\xi\sqrt{m}}{f(m)}c_{m-1} \qquad (27)$$

taking $c_m = d_m/[f(m)!\sqrt{m!}]$, then

$$\sqrt{1-|\xi|^2}\beta d_m = d_{m+1} - \xi m d_{m-1}. \qquad (28)$$

Putting $d_m = (-\frac{1}{2}\xi)^{m/2}F_m$, and by the help of the Hermite polynomial recurrence relation

$$2xH_n(x) = 2nH_{n-1}(x) + H_{n+1}(x) \qquad (29)$$

we obtain

$$|\xi_1,\beta_1\rangle_f = F_{01}\sum_{m=0}^{\infty} \frac{1}{f(m)!\sqrt{m!}}\left(-\frac{1}{2}\xi_1\right)^{m/2} H_m \times \left[\frac{\beta_1\sqrt{1-|\xi_1|^2}}{\sqrt{-2\xi_1}}\right]|m\rangle \qquad (30)$$

where the normalization factor has the form

$$|F_{01}|^{-2} = \left\{ \sum_{n=0}^{\infty} \frac{1}{[f(n)!]^2 n!} \left(-\frac{1}{2}|\xi_1|\right)^n \times \left| H_n \left[\frac{\beta_1 \sqrt{1-|\xi_1|^2}}{\sqrt{-2\xi_1}} \right] \right|^2 \right\}. \quad (31)$$

A similar expression is obtained for $|\xi_2, \beta_2\rangle_f$ but with $f(m)!$ replaced by $[f(m)!]^{-1}$. This is the definition of the NLSCS's depending on the form of function f when it is not a unitary operator. Note that it is formally similar to equation (21).

4 Non-classical properties

In this section, we shall evaluate the characteristic function (CF) and examine the correlation function, squeezing, and quadrature distributions. However before we proceed any further it is necessary to specify the nonlinearity function $f(n)$. From equation (30), it is clear that for every choice of $f(n)$ we shall get different NLSCS states. In the present case, we choose the following nonlinearity function which has been used in the description of the motion of a trapped ion [4], namely

$$f(n) = \frac{L_n^1(\eta^2)}{(n+1)L_n^0(\eta^2)}. \quad (32)$$

Where η is known as the Lamb-Dicke parameter and $L_n^m(x)$ are associated Laguerre polynomials given by

$$L_m^\sigma(x) = \sum_{s=0}^{m} \binom{m+\sigma}{m-s} \frac{(-x)^s}{s!}. \quad (33)$$

Clearly $f(n) = 1$ when $\eta = 0$ in this case the states of (21) and (30) become the standard squeezed coherent states. However, when $\eta \neq 0$ nonlinearity starts developing, with the degree of nonlinearity depending on the magnitude of the parameter η. The function $f(n)$ can be tailored at well by using $N-$lasers to engineer the Hamiltonian for nonlinear multiquanta JCM as proposed in [21].

To begin with the state (30) will be written in the form

$$|\xi, \beta\rangle_f = \sum_{n=0}^{\infty} B_n(\xi, f, \beta)|n\rangle, \quad (34a)$$

where

$$B_n(\xi, f, \beta) = F_0 \frac{(-\frac{1}{2}\xi)^{n/2}}{f(n)!\sqrt{n!}} H_n \left[\frac{\beta\sqrt{1-|\xi|^2}}{\sqrt{-2\xi}} \right]. \quad (34b)$$

For simplicity we use $B_n(\xi, f, \beta) = B_n$ in our calculations. It is clear that the coefficients B_n in general depend on polynomials, therefore we shall resort to perform numerical calculations.

4.1 Moments and the auto-correlation function $g^{(2)}(0)$

The average values of the annihilation and creation operators are derived by differentiating the characteristic function (CF) with respect to λ and $-\lambda^*$, respectively. The s-parameterized characteristic function $C(\beta, s)$ is defined by

$$C(\lambda, s) = Tr[\rho D(\lambda)] \exp\left(\frac{s}{2}|\lambda|^2\right) \quad (35)$$

with $D(\lambda)$ as given before. Now we calculate the CF for the state $|\xi, \beta\rangle_f$ equation (30). The density operator corresponding to the state $|\xi, \beta\rangle_f$ is

$$\rho = |\xi, \beta\rangle_{f\,f}\langle\beta, \xi| \quad (36)$$

then after some operators algebra, we can write the CF on the form,

$$C(\lambda, s) = \begin{cases} \exp\{-\frac{1}{2}(1-s)|\lambda|^2\} \sum_{m,n=0}^{\infty} B_n^* B_m \\ \quad \times \sqrt{\frac{m!}{n!}} (\lambda)^{n-m} L_m^{n-m}(|\lambda|^2), \quad n > m \\ \exp\{-\frac{1}{2}(1-s)|\lambda|^2\} \sum_{m,n=0}^{\infty} B_n^* B_m \\ \quad \times \sqrt{\frac{n!}{m!}} (-\lambda^*)^{m-n} L_n^{m-n}(|\lambda|^2), \quad m > n \end{cases} \quad (37)$$

where $L_m^\sigma(x)$ is the associated Laguerre polynomial. A hierarchy of observable conditions for a quantum state to be nonclassical have been derived in terms of experimentally accessible characteristic functions of quadratures [17]. Thus, the CF is obtained; and from it we can calculate any expectation value for the field operators. The s-ordered average value of a and a^+ can be calculated in the following way

$$\langle [a^{+k} a^l]_s \rangle = \mathrm{Tr}[\rho\{a^{+k} a^l\}_s] = \frac{\partial^k}{\partial\lambda^k} \frac{\partial^l}{\partial(-\lambda^*)^l} C(\lambda, s)|_{\lambda=\lambda^*=0}. \quad (38)$$

Or through an integration involving the s-parameterized quasiprobability function [1]. To calculate the moments of the quadratures in our state, one has to find average values of products of the operators a and a^+ in these states, on the form

$$\langle a^q \rangle = \sum_{s,r=0}^{\infty} B_r B_s^* \sqrt{\frac{r!}{(r-q)!}} \delta_{s,r-q} = \langle a^{+q} \rangle^* \quad (39)$$

$$\langle a^{+p} a^q \rangle = \sum_{s=p,r=q}^{\infty} B_r B_s^* \sqrt{\frac{r!}{(r-q)!}} \sqrt{\frac{s!}{(s-p)!}} \delta_{s-p,r-q}. \quad (40)$$

Practically applicable criteria for the nonclassicality of quantum states are formulated in terms of different types of the moments of creation and annihilation operators [17]. A method for measuring general space-time dependent correlation functions of quantized radiation fields have been proposed in reference [18]. It is shown that all the required moments can be determined by homodyne correlation measurements [17]. We look at one way to characterize nonclassicality behaviour namely the auto-correlation function.

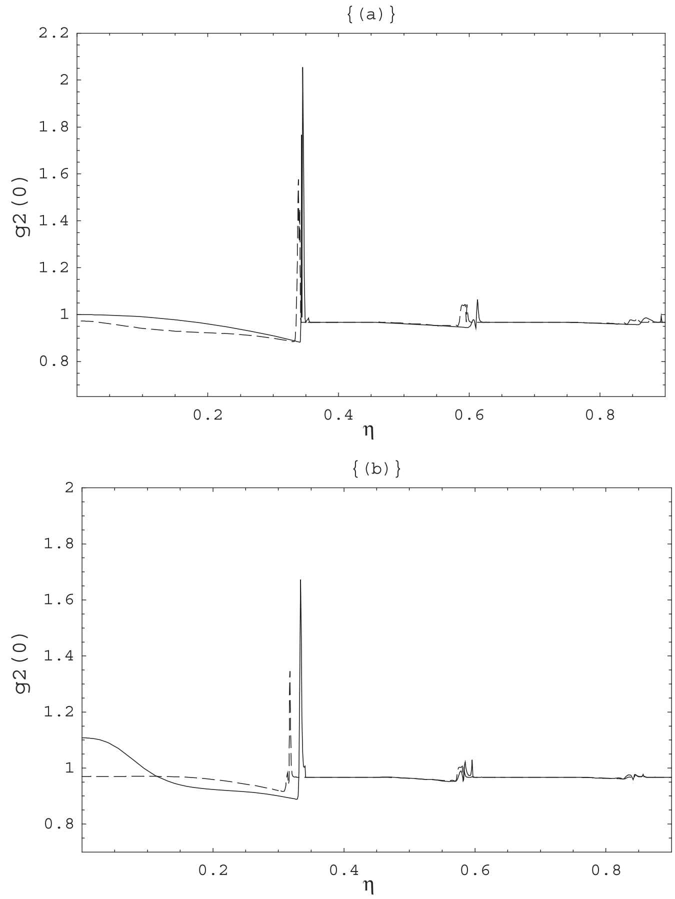

Fig. 1. Coherence function $g^{(2)}$ measured on vertical axis and horizontal axis indicates the nonlinearity parameter η. The direction of squeezing is zero $\phi = 0$, and (a) $\beta = 3$, $\xi = -\tanh r = 0$, i.e., $r = 0$, (solid curve) and $\beta = 3$, $\xi = -\tanh 1$, i.e., $r = 1$ (dashed curve); (b) $\beta = 3$, $r = 2$, (solid curve) and $\beta = 5$, and $r = 1$ (dashed curve).

The Glauber second-order coherence function is defined by

$$g^{(2)} = \frac{\langle a^{+2} a^2 \rangle}{\langle a^+ a \rangle^2}. \tag{41}$$

The light with $g^{(2)} < 1$ is a sub-Poissonian light, with $1 < g^{(2)} < 2$ is a super-Poissonian light, and with $g^{(2)} > 2$ is called super thermal light. Coherent light has $g^{(2)} = 1$ while thermal light has $g^{(2)} = 2$.

We plot the auto-correlation function $g^{(2)}$ in Figure 1 against the squeeze parameter ξ. We assume the parameters as follows: the direction of squeezing is taken to be zero, and (a) $\beta = 3$, and $\xi = 0$, $r = 0, 1$ (b) $\beta = 3, 5$, and $r = 1, 2$.

From Figure 1a sub-Poissonian light exists for $r = 0$ and $\eta > 0.1$ and the super-thermal exists for small β with $r > 2$ and super-Poissonian for $\eta = 0.4$. For increasing r the super-thermal behaviour is persistent. Also, when the displacement parameter β is increased the Poissonian behaviour is persistent. From Figure 1b, we note that the sub-Poissonian light exists for large range of η with higher values of squeezing and coherence parameters. The super-Poissonian light exists only with $r > 0.75$. The super-thermal statistics exists as r increases.

Generally the contribution of the nonlinearity parameters appears when the values of squeezing and coherence parameters are small. In this case we have examined the behaviour of sub-Poissonian, which depends originally on the squeezing and coherence parameters.

4.2 Photon number distribution

We begin by looking at the photon number distribution for the state $|\xi, \beta\rangle_f$. The photon number distribution $P(l)$ is

$$\begin{aligned} P(l) &= |\langle l|\xi, \beta\rangle_f|^2 \\ &= |B_l|^2. \end{aligned} \tag{42}$$

In Figure 2 we illustrate the photon number distribution $P(n)$ with $\beta = 3$, $\xi = -\tanh 1$, i.e., $r = 1$ and for different values of the nonlinearity parameter η. For $\eta = 0$ the photon number distribution has a maximum around the value of $|\beta|^2 + \sinh^2 r$. The photon number distribution has less than that value for $0 < \eta < 0.25$. Some oscillations of $P(l)$ appear for the values of $\eta > 0$. The oscillations reoccur later for higher values of $\eta > 0.2$ for lower values magnitude of photon number.

4.3 Squeezing

We next look at squeezing. To do this, we introduce the two quadrature operators

$$X_1 = \frac{1}{2}(a + a^+), \qquad X_2 = \frac{1}{2i}(a - a^+). \tag{43}$$

These are dimensionless position and momentum operators for a harmonic oscillator. Their commutator is $[X_1, X_2] = i/2$ and they satisfy the uncertainty relation $\langle(\Delta X_1)^2\rangle\langle(\Delta X_2)^2\rangle \geq 1/16$ with the variance $\langle(\Delta X_j)^2\rangle = \langle X_j^2\rangle - \langle X_j\rangle^2$. The field is said to be squeezed if $(\Delta X_j)^2 < 1/4$ for ($j = 1$ or 2).

The average values of the quadrature field operators $\langle X_1\rangle$ and $\langle X_2\rangle$ are directly computed. Also variances of the quadrature field operators $\langle(\Delta X_1)^2\rangle$ and $\langle(\Delta X_2)^2\rangle$ are computed.

The squeezing is best parameterized by the following parameters

$$q_j = \frac{\langle(\Delta X_j)^2\rangle - 0.25}{0.25}, \qquad j = 1, 2 \tag{44}$$

such that squeezing exits for $-1 < q_j < 0$, i.e., the squeezing condition now reads $q_j < 0$, and the maximum squeezing corresponds to $q_j = -1$. Squeezing in one quadrature is achieved at the expense of increase noise in the conjugate quadrature. Therefore, if one of q_j's is less than zero, then the other should be greater than zero.

In Figure 3 we plot q_1 against η with $\beta = 0.5$, $\phi = 0$, and $r = 0.25$. Squeezing for increasing r is shown as may

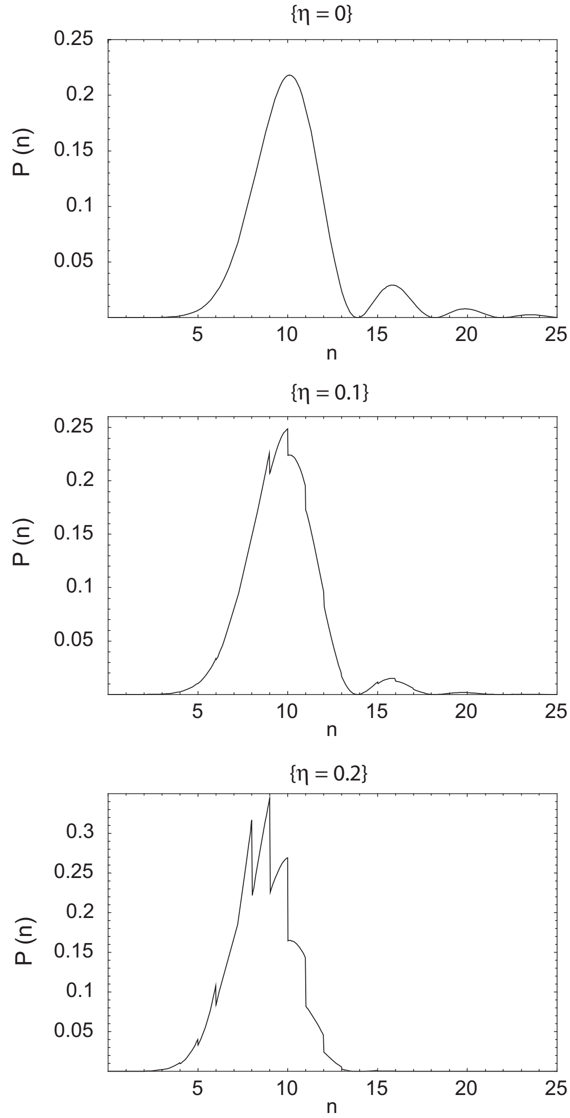

Fig. 2. The photon number distribution $P(n)$ with $\beta = 3$, $\phi = 0$, and $r = 1$, for values of $\eta = 0, 0.1, 0.2$.

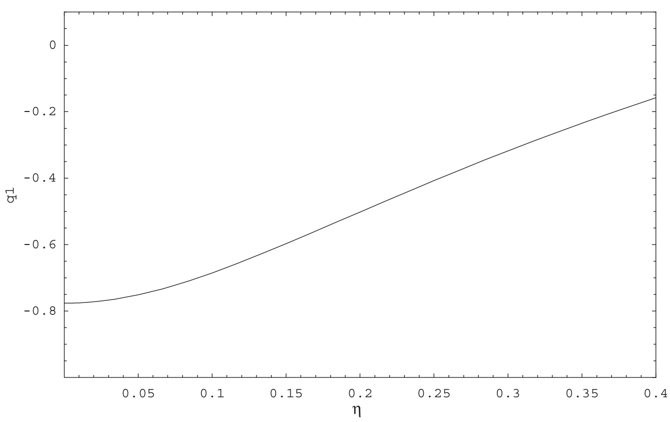

Fig. 3. Squeezing parameter q_1 against the nonlinearity parameter η, with $\beta = 0.5$ and $r = 0.25$.

be expected. The quadrature operator q_2 has no squeezing for these states for the same parameters. The squeezing behaviour decreases for increase of the coherence parameter β, and becomes highly effective for $r > 0$ and small of β. After $\eta = 0.4$ the behaviour becomes remarkably different and larger values of squeezing are observed [25].

Numerical calculations show that with increasing Lamb-Dicke parameter η the nonclassical effects are typically decreasing for the range $0 < \eta < 0.45$. But with increasing η, i.e., $\eta > 0.45$, the behavior of the NLSCS's is reversed, i.e., the nonclassical effects are typically increased. Therefore, increasing values of η (more than 0.45) result in significant oscillations as shown in Figure 2.

4.4 Quadrature distributions

In order to calculate the quadrature component distribution for the NLSCS state (i.e., the phase-parameterized field strength distribution) we write

$$P(x, \Phi) = |\langle x, \Phi|\xi, \beta\rangle_f|^2 \tag{45}$$

which can be measured in balanced homodyne detection. We first expand the eigenstate $|x, \Phi\rangle$ of quadrature component

$$x(\Phi) = \frac{1}{\sqrt{2}}(e^{-i\Phi}a + e^{i\Phi}a^+) \tag{46}$$

with eigenvalue x in the photon number basis as [26]

$$|x, \Phi\rangle = \frac{1}{\pi^{\frac{1}{4}}} \exp(-\frac{1}{2}x^2) \sum_{j=0}^{\infty} \frac{e^{i\Phi\, j}}{\sqrt{2^j j!}} H_j(x)|j\rangle. \tag{47}$$

We have the quadrature component distribution (3.6) in the form

$$P(x, \Phi) = \frac{1}{\pi^{\frac{1}{2}}} \exp(-x^2) \sum_{j,l=0}^{\infty} \frac{\cos[\Phi(l-j)]}{\sqrt{2^{(l+j)} j! l!}} \times B_j^* B_l H_j(x) H_l(x). \tag{48}$$

In Figure 4 we plot the phase-parameterized field strength distribution (quadrature component) $P(x, \Phi)$ with $\beta = 3$, $r = 1$ and (A) $\eta = 0$, (B) $\eta = 0.2$. In general the figures for $P(x, \Phi)$ are symmetric around $\Phi = 0$. Changing η does not make a remarkable difference for the distributions except of the higher magnitude of values. For $x = 0$ it is observed that two peak shape for the distributions about $\Phi = \pm\pi/2$, but all phase information disappears as x takes values larger than 3.5. However, when $x > 2.5$, we show that the two-peak shape for the distribution submerge and a single broad peak is shown as Φ gets closer to 0.

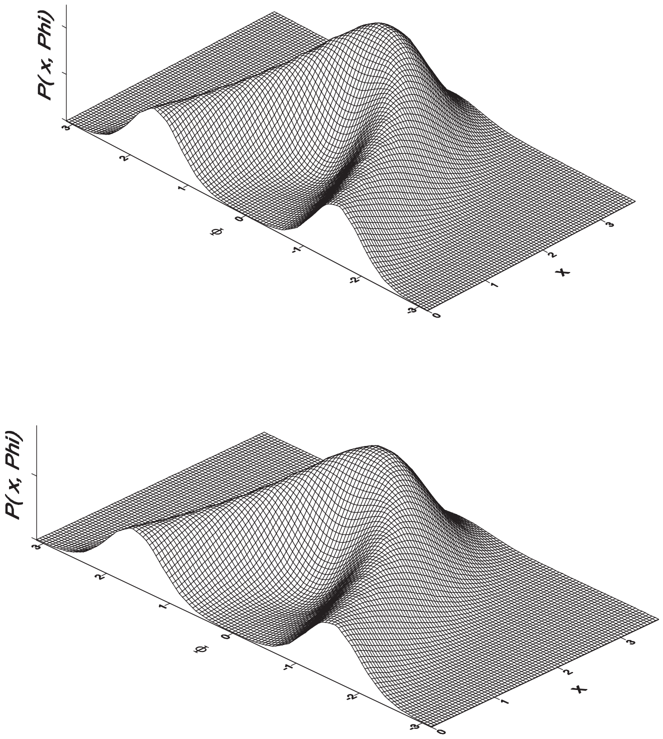

Fig. 4. Plots of the phase-parameterized field strength distribution (quadrature component) $P(x,\Phi)$ with, $\beta = 3$, $r = 1$, $\phi = 0$, and (A) $\eta = 0$, (B) $\eta = 0.2$.

5 Generation scheme

After the discussion of the properties of the NLSCS's, we wish to consider the production of such states.An ion confined in an electromagnetic trap can be regarded as a particle with a quantized centre-of-mass motion in harmonic potential. The changes in motion of an ion can be controlled by exciting or deexcitting the internal states of the ion by a classical laser field. The ion's motion is altered since stimulated emission and absorption are always accompanied by momentum exchange between the laser field and the ion. If the vibrational amplitude of the ion is much smaller than the laser wave-length (the Lamb-Dicke limit [20]) and if the driving field is tuned to one of the vibrational sidebands the model simplifies to a form similar to the Jaynes-Cummings model [27] where the radiation field is replaced by the vibrational motion of the ion. The dissipative effects of the cavity may be neglected in this model as the coupling between the vibrational modes and the environment is extremely weak [20–23].

We have the state (25) $(C_i|\xi_i, \beta_i\rangle_f = \beta_i|\xi_i, \beta_i\rangle_f)$ where C_i are given by (12). We look for a generation scheme for such states. In experiments with trapped atoms, the electronic transitions are accompanied by annihilations and creations of centre-of-mass motional quanta. We use the trapped atom technique as a framework for the generation of such states. It has been proposed [21,22] to use a number of laser fields of the same frequency but with different amplitudes and different Lamb-Dicke parameters, in order to engineer the photon-number depending coupling. Thus to implement this scheme we apply N_1 laser fields with the same frequency tuned lower than the electronic transition frequency by a single frequency of centre-of-atom quantized motion, and N_2 laser fields with the same frequency tuned to the blue-sideband of the transition frequency by the same frequency, besides N_3 laser fields of the carrier type. The Hamiltonian for the trapped atom of transition frequency ω_0, and the centre-of-mass quantized motion of frequency ω and annihilation and creation operators a and a^+ respectively, in interaction with the different laser fields is given by

$$\begin{aligned}\mathbf{H} &= \omega a^+ a + \frac{\omega_0}{2}\sigma_z \\ &+ \vec{\mu}\cdot\Bigg\{\sum_{r=1}^{N_1} \vec{E}_r \exp[ik_r x - (\omega_0 - \omega)t + i\phi_r] \\ &+ \sum_{l=1}^{N_2} \vec{E}_l \exp[ik_l x - (\omega_0 + \omega)t + i\phi_l] \\ &+ \sum_{u=1}^{N_3} \vec{E}_u \exp[ik_u x - \omega_0 t + i\phi_u]\Bigg\}. \qquad (49)\end{aligned}$$

The quantized centre-of-mass position $\hat{x}$ is given by $\hat{x} = \Delta x(a + a^+)$ where Δx is the standard deviation of $\hat{x}$ in the ground state of the harmonic potential. In equation (49) σ_z is the atomic inversion operator, $\hat{\vec{\mu}}$ the atomic dipole moment operator and can be written in the form $\hat{\vec{\mu}} = \mu_{0s}(\sigma_+ + \sigma_-)$ where $\sigma_+(\sigma_-)$ is the raising (lowering) operator for the atomic states. With $\vec{E}_i$ the laser amplitudes, and ϕ_i their phases. Rotating wave approximation is then used and neglecting terms with fast oscillations then we get the Hamiltonian (49) in the following simplified form:

$$\mathbf{H}_{\mathrm{int}} = \sigma_+\Big\{\Omega_{L_1} f_1(N) a - \Omega_{L_2} a^+ f_2(N) + \Omega_{L_3} f_3(N)\Big\} + h.c. \qquad (50a)$$

where

$$f_1(N) = i\sum_{r=1}^{N_1}\sum_{m=1}^{\infty} \frac{\Omega_r \eta_r e^{i\phi_r}}{\Omega_{L_1}} \frac{(-\eta_r^2)^m}{m!(m+1)!} \frac{N!}{(N-m)!} e^{-\eta_r^2} \qquad (50b)$$

$$f_2(N) = i\sum_{l=1}^{N_2}\sum_{m=1}^{\infty} \frac{\Omega_l \eta_l e^{i\phi_l}}{\Omega_{L_2}} \frac{(-\eta_l^2)^m}{m!(m+1)!} \frac{N!}{(N-m)!} e^{-\eta_l^2} \qquad (50c)$$

$$f_3(N) = \sum_{u=1}^{N_3}\sum_{m=1}^{\infty} \frac{\Omega_u e^{i\phi_u}}{\Omega_{L_3}} \frac{(-\eta_u^2)^m}{(m!)^2} \frac{N!}{(N-m)!} e^{-\eta_u^2} \qquad (50d)$$

where Ω_s is the electronic Rabi frequencies associated with the field amplitude E_s and Ω_{L_i} characteristic Rabi frequencies. As it has been shown [21] the amplitudes (the geometry) and phases of the laser fields can be controlled to produce a photon-number dependent function tailored at well.

The master equation for the density matrix under spontaneous emission with energy dissipation rate γ is given by

$$\frac{\partial \rho}{\partial t} = -i[\mathbf{H}_{\text{int}}, \rho] + \frac{\gamma}{2}(2\sigma_- \rho \sigma_+ - \sigma_+ \sigma_- \rho - \rho \sigma_+ \sigma_-). \quad (51)$$

The stationary solution $\bar{\rho}_s$ for this master equation is obtained by setting $\partial\rho/\partial t$. A solution for the density operator $\bar{\rho}_s$, may be given as

$$\bar{\rho}_s = |g\rangle|\xi\rangle\langle\xi|\langle g| \quad (52)$$

with $|g\rangle$ the electronic ground state of the atom ($\sigma_-|g\rangle = 0, \langle g|\sigma_+ = 0$) and the state $|\xi\rangle$ is given from

$$\mathbf{H}_{\text{int}}|g\rangle|\xi\rangle = 0. \quad (53)$$

This turns out to be

$$\Omega_{L_1} f_1(N)a - \Omega_{L_2} a^+ f_2(N) + \Omega_{L_0} f_3(N)|\xi\rangle = 0. \quad (54)$$

This can be cast in the form of the NLSCS by taking

$$\begin{aligned} f(N) &= f_3^{-1}(N-1)f_1(N-1), \\ f^{-1}(N) &= f_3^{-1}(N-1)f_2(N), \\ \xi &= \frac{\Omega_{L_2}}{\Omega_{L_1}} \quad \text{and} \quad \beta\sqrt{1-|\xi|^2} = -\frac{\Omega_0}{\Omega_{L_1}}. \end{aligned} \quad (55)$$

Then the form of equation (54) tend to equation (25), thus leading to the realization of NLSCS's.

6 Conclusions

In this article we have studied a nonlinear extension of the single-mode squeezed vacuum and squeezed coherent states. Some basic definitions and properties of SU(1,1) Lie algebra have been considered. Various applications of these results in the context of the two-photon realization of SU(1,1) in quantum optics are also considered. The NLSS's realization of SU(1,1) Lie group have been constructed. We have defined the nonlinear squeezed coherent states. We have discussed numerically the properties of these states. In particular, the photon number distribution and squeezing. Several moments have been calculated. The second-order correlation function $g^{(2)}$ has been investigated numerically and shown that the NLSCS's exhibit sub-Poissonian behaviour. We have analyzed the quadrature component distributions for these states and have presented analytical and numerical results. A generation scheme for NLSCS has been presented. Recently, some different classes of nonlinear squeezed coherent states are discussed in reference [28].

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